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Algebraic approach to q-deformed supersymmetric variants of the Hubbard model with pair hoppings

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Abstract: We construct two quantum spin chains hamiltonians with quantum $sl(2|1)$ invariance. These spin chains define variants of the Hubbard model and describe electron models with pair hoppings. A cubic algebra that admits the Birman–Wenzl–Murakami algebra as a quotient allows exact solvability of the periodic chain. The two hamiltonians, respectively built using the distinguished and the fermionic bases of $\mathcal{U}_q(sl(2|1))$ differ only in the boundary terms. They are actually equivalent, but the equivalence is non local. Reflection equations are solved to get exact solvability on open chains with non trivial boundary conditions. Two families of diagonal solutions are found. The centre and the scasimirs of the quantum enveloping algebra of $sl(2|1)$ appear as tools for the construction of exactly solvable hamiltonians.

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Contents

1 Introduction

Since a few years, there is a considerable interest about some models of strongly correlated electrons, in particular those of the families of the t–J model and of the Hubbard model. The reason is that they exhibit some very interesting physical properties related with high T_c superconductivity. Among these models, some have the property of supersymmetry, or quantum supersymmetry. This is the case for some generalisations of the t–J model. It is also the case for some variants of the Hubbard models in which a pair hopping term is included ([1, 2, 3] and [4] for quantum supersymmetry).

The aim of this paper is to present the construction of two variants of the supersymmetric Hubbard model with pair hoppings, to describe the algebra that ensures their integrability and to solve the reflection equations which lead to integrable boundary terms.

From the expression of the series of Casimir operators $\mathcal{C}_p$ of $\mathcal{U}_q(sl(2|1))$, we derive quantum spin chain hamiltonians $\mathcal{H}$ with built-in $\mathcal{U}_q(sl(2|1))$ invariance:

$$\mathcal{H} = \sum_{i=1}^{L-1} 1 \otimes \cdots \otimes \underbrace{(\rho \otimes \rho)\Delta(\mathrm{Pol}\{\mathcal{C}_p\})}_{\text{sites } i,i+1} \otimes \cdots \otimes 1 \ . \tag{1.1}$$

An ingredient of the construction is the knowledge of a series of Casimir operators of the quantum algebra. We will also see that the knowledge of scasimirs (given in section 2) leads to some exactly solvable hamiltonians.

Another ingredient of the construction is the four dimensional one parameter typical representation ρ of $\mathcal{U}_q(sl(2|1))$, so that the hamiltonian describes a four states per site spin chain with two parameters (the parameter of the representation together with the deformation parameter q).

The integrability of the closed chain is based on the algebra

$$(b_i + q)\left(b_i - q\lambda^2\right)\left(b_i - q^{-1}\lambda^{-2}\right) = 0 \ , \tag{1.2}$$

$$b_i b_{i\pm1} b_i = b_{i\pm1} b_i b_{i\pm1} \ , \tag{1.3}$$

$$b_i b_j = b_j b_i \qquad \text{for} \qquad |i-j| \geq 2 \ , \tag{1.4}$$

$$(b_i - x)b_{i\pm1}^{-1}(b_i - x) - b_i^{-1}(b_{i\pm1} - x)b_i^{-1} = (b_{i\pm1} - x)b_i^{-1}(b_{i\pm1} - x) - b_{i\pm1}^{-1}(b_i - x)b_{i\pm1}^{-1} \ . \tag{1.5}$$

This algebra was proved in [5] to be sufficient to construct a solution $\check{\mathcal{R}}(u)$ of the Yang–Baxter algebra (see below (4.4)). Moreover, the Birman–Wenzl–Murakami algebra [6, 7] is a quotient of this algebra. Our realisation of the algebra (1.2–1.5) actually does not satisfy the supplementary relations of the BWM algebra. The operators b_i enters in the expression of the two site hamiltonian as

$$\mathcal{H}_{i,i+1} = b_i - b_i^{-1} \ . \tag{1.6}$$

A remarkable fact is that, using the distinguished and fermionic bases of $\mathcal{U}_q(sl(2|1))$, we obtain two different hamiltonians, the difference being in the boundary terms. The same phenomenon was described in [8] with three state per site spin chains (deformed supersymmetric t–J model). These hamiltonians are actually equivalent *on open chains*, but this equivalence, which comes from a Reshetikhin twist, is non trivial since it is non local on the chain.

One of the hamiltonians (constructed with the distinguished basis) was known to be exactly solvable [9, 4]. It was obtained in [4], starting from the expression of the spectral parameter $\mathcal{R}$-matrix of $\mathcal{U}_q(sl(2|1))$.

The reflection equations associated with the solution $\check{\mathcal{R}}(u)$ of the Yang–Baxter algebra are solved for diagonal $\mathcal{K}$ matrices. Two families of one parameter solutions are found for each equation, leading to four possible boundary terms for exactly solvable open chain hamiltonians. This number of solutions is the same as found in [10] in the case of the supersymmetric t–J model. It is then shown that a special choice of these boundary terms is exactly the difference of the two hamiltonians built from the distinguished and the fermionic bases.

In the Appendix, the expressions of the scasimir operators of the (non quantized) $sl(2|1)$ superalgebra are given.

This work was already completed when the paper [11] appeared. In this paper, the hamiltonian (5.6) corresponding to the distinguished basis is studied. One of the solutions (i.e. 6.5) for the reflection equations is given and the corresponding integrable boundary terms are computed. The Bethe ansatz equations are also written. Analogous results were also obtained in [12] for the same model with isotropy. Similar studies also exist for eight-state $\mathcal{U}_q(sl(3|1))$-invariant models [13, 14].

2 The quantum algebra $\mathcal{U}_q(sl(2|1))$

2.1 Definitions

The superalgebra $\mathcal{U}_q(sl(2|1))$ in the distinguished basis is the associative superalgebra over $\mathbb{C}$ with generators $k_i^{\pm 1}$, e_i, f_i, $(i = 1, 2)$ and relations

$$
\begin{aligned}
&k_1 k_2 = k_2 k_1 \ , \\
k_i e_j k_i^{-1} &= q^{a_{ji}} e_j \ , \qquad & k_i f_j k_i^{-1} &= q^{-a_{ji}} f_j \ , \\
e_1 f_1 - f_1 e_1 &= \frac{k_1 - k_1^{-1}}{q - q^{-1}} \ , \qquad & e_2 f_2 + f_2 e_2 &= \frac{k_2 - k_2^{-1}}{q - q^{-1}} \ , \\
[e_1, f_2] &= 0 \ , \qquad & [e_2, f_1] &= 0 \ , \\
&e_2^2 = f_2^2 = 0 \ , \\
&e_1^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0 \ , \\
&f_1^2 f_2 - (q + q^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0 \ .
\end{aligned}
\tag{2.1}
$$

The matrix (a_{ij}) is the distinguished Cartan matrix of $sl(2|1)$, i.e.

$$
(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}
\tag{2.2}
$$

The $\mathbb{Z}_2$-grading in $\mathcal{U}_q(sl(2|1))$ is uniquely defined by the requirement that the only odd generators are e_2 and f_2, i.e.

$$
\begin{aligned}
&\deg\,(k_i) = \deg\,(k_i^{-1}) = 0 \ , \\
&\deg\,(e_1) = \deg(f_1) = 0 \ , \\
&\deg\,(e_2) = \deg(f_2) = 1 \ .
\end{aligned}
\tag{2.3}
$$

We define a Hopf algebra structure on $\mathcal{U}_q(sl(2|1))$ by

$$
\begin{aligned}
\Delta(k_i) &= k_i \otimes k_i \ , \\
\Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i \ , \\
\Delta(f_i) &= f_i \otimes k_i^{-1} + 1 \otimes f_i \ ,
\end{aligned}
\tag{2.4}
$$

2.2 Centre and scasimirs

In the enveloping algebra $\mathcal{U}_q(sl(2|1))$, we define for $p \in \mathbb{Z}$ the elements

$$\mathcal{Q}^{(+)}{}_p = k_1^{2p-1} k_2^{4p-2} \Big\{ [h_1+h_2+1][h_2] - f_1 e_1 - f_2 e_2 [h_1+h_2+1] - f_3 e_3 [h_2-1] + q^{-1} f_3 e_2 e_1 k_2 + q f_1 f_2 e_3 k_2^{-1} + \left(1+q^{2-4p}\right) f_2 f_3 e_3 e_2 \Big\} , \qquad (2.5)$$

and

$$\mathcal{Q}^{(-)}{}_p = k_1^{2p-1} k_2^{4p-2} q^{-2p} \Big\{ q f_2 e_2 [h_1+h_2] + q f_3 e_3 [h_2-2] - q^{-1} f_3 e_2 e_1 k_2 - q^3 f_1 f_2 e_3 k_2^{-1} - \left(1+q^2\right) f_2 f_3 e_3 e_2 \Big\} , \qquad (2.6)$$

where

$$e_3 = e_1\, e_2 - q^{-1} e_2\, e_1 \qquad \text{and} \qquad f_3 = f_2\, f_1 - q\, f_1\, f_2 \ . \qquad (2.7)$$

The operators $\mathcal{Q}^{(\pm)}$ satisfy the following set of relations

$$\mathcal{Q}^{(+)}_{p_1} \mathcal{Q}^{(-)}_{p_2} = \mathcal{Q}^{(-)}_{p_1} \mathcal{Q}^{(+)}_{p_2} = 0 \qquad \forall\ p_1, p_2 \in \mathbb{Z} \ , \qquad (2.8)$$

$$\mathcal{Q}^{(+)}_{p_1} \mathcal{Q}^{(+)}_{p_2} = \mathcal{Q}^{(+)}_{p_3} \mathcal{Q}^{(+)}_{p_4} \qquad \text{if} \qquad p_1 + p_2 = p_3 + p_4 \ , \qquad (2.9)$$

$$\mathcal{Q}^{(-)}_{p_1} \mathcal{Q}^{(-)}_{p_2} = \mathcal{Q}^{(-)}_{p_3} \mathcal{Q}^{(-)}_{p_4} \qquad \text{if} \qquad p_1 + p_2 = p_3 + p_4 \ . \qquad (2.10)$$

In the enveloping algebra $\mathcal{U}_q(sl(2|1))$, there are two abelian subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$, generated respectively by the operators $\mathcal{Q}^{(+)}$ and $\mathcal{Q}^{(-)}$. They are such that

$$\forall x^+ \in \mathcal{A}^{(+)}, \quad \forall x^- \in \mathcal{A}^{(-)}, \qquad x^+ x^- = x^- x^+ = 0 \ . \qquad (2.11)$$

The elements $\mathcal{Q}^{(\pm)}$ allow us to build generators of the centre of $\mathcal{U}_q(sl(2|1))$, and also a set of scasimirs: if we define, for $p \in \mathbb{Z}$

$$\mathcal{C}_p = \mathcal{Q}^{(+)}_p + \mathcal{Q}^{(-)}_p \ , \qquad (2.12)$$

$$\mathcal{S}_p = \mathcal{Q}^{(+)}_p - \mathcal{Q}^{(-)}_p \ , \qquad (2.13)$$

then

- 1 and the $\mathcal{C}_p$, for $p \in \mathbb{Z}$, generate the centre of $\mathcal{U}_q(sl(2|1))$, (for q not a root of unity):

  $$x \mathcal{C}_p - \mathcal{C}_p x = 0 \qquad \forall x \in \mathcal{U}_q(sl(2|1)) \ . \qquad (2.14)$$

  (See [8, 15], and [16] for the non quantized case).

- The $\mathcal{S}_p$ commute with the bosonic elements of $\mathcal{U}_q(sl(2|1))$ and anticommute with the fermionic ones (although they are themselves bosonic)

  $$\mathcal{S}_p x - (-1)^{\deg(x)} x \mathcal{S}_p = 0 \quad (\forall x \in \mathcal{U}_q(sl(2|1)) \text{ with homogeneous degree}) \ . \qquad (2.15)$$

Furthermore, the $\mathcal{C}_p$, $\mathcal{S}_p$ obey the set of relations

$$\mathcal{C}_{p_1}\mathcal{C}_{p_2} = \mathcal{C}_{p_3}\mathcal{C}_{p_4} \qquad \text{if} \qquad p_1+p_2=p_3+p_4 \; , \tag{2.16}$$
$$\mathcal{C}_{p_1}\mathcal{C}_{p_2} = \mathcal{S}_{p_3}\mathcal{S}_{p_4} \qquad \text{if} \qquad p_1+p_2=p_3+p_4 \; , \tag{2.17}$$
$$\mathcal{C}_{p_1}\mathcal{S}_{p_2} = \mathcal{S}_{p_3}\mathcal{C}_{p_4} \qquad \text{if} \qquad p_1+p_2=p_3+p_4 \; , \tag{2.18}$$

which is equivalent to the set (2.8, 2.9, 2.10). Relation (2.16) was given in [16] for the non quantized case and in [8] in the quantized case.

In particular, on representations on which $\mathcal{C}_p$ are different from 0, the quotient $\frac{\mathcal{S}_p}{\mathcal{C}_p}$ plays the role of $(-1)^F$, i.e.:

$$\left(\frac{\mathcal{S}_p}{\mathcal{C}_p}\right)^2 = 1 \; , \tag{2.19}$$
$$\frac{\mathcal{S}_p}{\mathcal{C}_p}x - (-1)^{\deg(x)}x\frac{\mathcal{S}_p}{\mathcal{C}_p} = 0 \quad (\forall x \in \mathcal{U}_q(sl(2|1)) \text{ with homogeneous degree}) \, . \tag{2.20}$$

Most hamiltonians in the following will be constructed using $(\rho\otimes\rho)\Delta(\mathcal{C}_p)$, with

$$\begin{aligned}\mathcal{C}_p = k_1^{2p-1}k_2^{4p-2} \; \Big\{ & [h_1+h_2+1][h_2] - f_1e_1 + f_2e_2([h_1+h_2]q^{1-2p} - [h_1+h_2+1]) \\ & + f_3e_3([h_2-2]q^{1-2p} - [h_2-1]) + (q-q^{-1})q^{-1-p}[p]f_3e_2e_1k_2 \\ & + (q-q^{-1})q^{2-p}f_1f_2e_3k_2^{-1}[p-1] + \\ & + (q-q^{-1})^2q^{1-2p}[p][p-1]f_2f_3e_3e_2\Big\} \, . \end{aligned} \tag{2.21}$$

2.3 Four dimensional representation

We use the one-parameter four-dimensional representation, acting on the vector space V of dimension 4 and defined (in the distinguished basis) by

$$\begin{aligned}\rho(e_1) &= -\omega q E_{23} \\ \rho(e_2) &= (\lambda-\lambda^{-1})E_{12} + (q\lambda - q^{-1}\lambda^{-1})E_{34} \\ \rho(f_1) &= -q^{-1}E_{32} \\ \rho(f_2) &= E_{21}+E_{43} \\ \rho(k_1) &= \lambda^{-1}\left(E_{11}+E_{22}+q^{-1}E_{33}+q^{-1}E_{44}\right) \\ \rho(k_2) &= \omega\lambda^{-2}\left(E_{11}+q^{-1}E_{22}+q^{-1}E_{33}+q^{-2}E_{44}\right) \; ,\end{aligned} \tag{2.22}$$

where $\omega=\pm1$ is a discrete parameter that allows two different (inequivalent) representations for each value of the continuous parameter $\lambda\equiv q^\mu$ [15]. The discrete parameter ω is a remnant of the quantisation of the value of k_1 on the highest weight vector in finite dimension.

The E_{ij} are the standard elementary matrices of End(V) given by

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl} \; . \tag{2.23}$$

The operators are represented by ordinary matrices, with complex (commuting) elements. We do not consider supermatrices. The traces are not supertraces. Tensor products of representations are non-graded tensor products. We indeed use, as in [8], the non-graded coproduct defined from the usual graded one as (in Sweedler's notation)

$$\Delta^{n.g.}(x) = \sum x_{(1)} g^{\deg(x_{(2)})} \otimes x_{(2)} \qquad \text{where} \qquad \Delta(x) = \sum x_{(1)} \otimes x_{(2)} \;, \tag{2.24}$$

g being the diagonal element in $\text{End}(V)$ defined by $g = \sum_{j=1}^{\dim V} (-1)^{\deg(j)} E_{jj}$ with $\deg(1) = \deg(4) = 0$ and $\deg(2) = \deg(3) = 1$. This is nothing but a Jordan–Wigner transformation. Practically, on tensor products of representations, this amounts to the use of the graded coproduct Δ, the evaluation of the representations $\rho_1 \otimes \rho_2$ and then application of the transformation

$$E_{ij} \otimes E_{kl} \longrightarrow (-1)^{\deg(j)(\deg(k)+\deg(l))} E_{ij} \otimes E_{kl} \;. \tag{2.25}$$

In the following, this will be implicitly included in the construction. This use of ordinary matrices and non graded coproduct is actually equivalent to the standard procedure, and leads to the same conclusions. It is however sometimes simpler in actual computations.

The transformation from Δ to the non-graded $\Delta^{n.g.}$ was used by Majid to bosonize super Hopf algebras [17]. It is a simple case of transmutation. A transformation was also defined in [18] and applied to the $\mathcal{R}$-matrix, which allowed to consider non-graded Yang–Baxter equations.

3 Braid group representation

Explicit computation shows that

$$\begin{aligned}(\rho \otimes \rho)\Delta(\mathcal{C}_p) = -q^{-1}\lambda^{8p-4} \big([2\mu][2\mu+1]\mathcal{O}_0 + q^{2p-1}[2\mu][2\mu+2]\mathcal{O}_1 \\ + q^{4p-2}[2\mu+1][2\mu+2]\mathcal{O}_2 \big) \;,\end{aligned} \tag{3.1}$$

where the expression of the operators $\mathcal{O}_a$ is given later in equation (4.3).

The operators $\mathcal{O}_a$ satisfy the relations

$$\begin{aligned} \mathcal{O}_a \mathcal{O}_b &= \delta_{a,b} \mathcal{O}_a \\ \mathcal{O}_0 + \mathcal{O}_1 + \mathcal{O}_2 &= \text{Id} \;. \end{aligned} \tag{3.2}$$

The operators $\mathcal{O}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$ are actually projectors on the representations of dimension 4, 8 and 4, respectively, that enter in the decomposition of the tensor product $\rho \otimes \rho$ (using Δ).

Inverting (3.1) allows us to express these projectors directly in terms of evaluations on the tensor product $\rho \otimes \rho$ of some Casimir operators $\mathcal{C}_p$:

$$\mathcal{O}_0 = \frac{q^4 \lambda^{-8p-4}}{[2\mu][2\mu+1](q^4-1)(q^2-1)} (\rho \otimes \rho)\Delta \left(-q^3 \lambda^8 \mathcal{C}_p + (q+q^{-1})\mathcal{C}_{p+1} - q^{-3}\lambda^{-8}\mathcal{C}_{p+2} \right) \;,$$

$$\mathcal{O}_1 = \frac{q^{-2p+4}\lambda^{-8p-4}}{[2\mu][2\mu+2](q^4-q^2)(q^2-1)}(\rho\otimes\rho)\times$$
$$\times\,\Delta\left(q^2\lambda^8\mathcal{C}_p-(q^2+q^{-2})\mathcal{C}_{p+1}+q^{-2}\lambda^{-8}\mathcal{C}_{p+2}\right)\ ,$$
$$\mathcal{O}_2 = \frac{q^{-4p+4}\lambda^{-8p-4}}{[2\mu+1][2\mu+2](q^4-q^2)(q^4-1)}(\rho\otimes\rho)\times$$
$$\times\,\Delta\left(-q\lambda^8\mathcal{C}_p+(q+q^{-1})\mathcal{C}_{p+1}-q^{-1}\lambda^{-8}\mathcal{C}_{p+2}\right)\ , \qquad (3.3)$$

where, again, $\lambda = q^\mu$.

As a consequence of (3.2), the algebra generated by all the $\mathcal{U}_q(sl(2|1))$ invariant operators $(\rho\otimes\rho)\Delta(\mathcal{C}_p)$ is $\mathrm{Vect}(\mathcal{O}_0,\mathcal{O}_1,\mathcal{O}_2) = \mathrm{Vect}(\mathrm{Id},\mathcal{O}_0,\mathcal{O}_2) \subset \mathrm{End}(V\otimes V)$. Within this algebra, we look for operators b satisfying the braid group relations

$$b_i b_{i\pm1} b_i = b_{i\pm1} b_i b_{i\pm1}\ , \qquad (3.4)$$
$$b_i b_j = b_j b_i \qquad \text{for} \qquad |i-j|\geq 2\ , \qquad (3.5)$$

where

$$b_i \equiv b_{i,i+1} = 1\otimes\cdots\otimes b\otimes\cdots\otimes 1\ , \qquad (3.6)$$

in which b occupies positions $i, i+1$.

We find two non trivial solutions to these equations, given by

$$b = -q\mathrm{Id} + q\lambda\frac{[2\mu]}{[\mu]}\mathcal{O}_0 + \lambda^{-1}\frac{[2\mu+2]}{[\mu+1]}\mathcal{O}_2\ , \qquad (3.7)$$

the other one being its inverse b^{-1}.

$$b^{-1} = -q^{-1}\mathrm{Id} + q^{-1}\lambda^{-1}\frac{[2\mu]}{[\mu]}\mathcal{O}_0 + \lambda\frac{[2\mu+2]}{[\mu+1]}\mathcal{O}_2\ . \qquad (3.8)$$

These are the only solutions for generic $\lambda = q^\mu$. For particular values of λ, i.e. $\lambda = \pm q^{-1/2}$ for instance, there are other solutions to the braid relations, which can lead to Temperley–Lieb algebra [19].

We define $x = (\lambda-\lambda^{-1})(q\lambda - q^{-1}\lambda^{-1})$ and $y = ([\mu][\mu+1])^{1/2} = x^{1/2}/(q-q^{-1})$, including the freedom for a sign in y.

The explicit expressions for b and b^{-1} are

$$\begin{aligned} b =\ & q\lambda^2 E_{11}\otimes E_{11} + (q\lambda^2-q)E_{11}\otimes E_{22} + (q\lambda^2-q)E_{11}\otimes E_{33} + xE_{11}\otimes E_{44} \\ & + q\lambda(E_{12}\otimes E_{21} + E_{21}\otimes E_{12}) + q^{-1/2}x^{1/2}(E_{12}\otimes E_{43} + E_{21}\otimes E_{34}) \\ & + q\lambda\omega(E_{13}\otimes E_{31} + E_{31}\otimes E_{13}) - q^{1/2}x^{1/2}\omega(E_{13}\otimes E_{42} + E_{31}\otimes E_{24}) \\ & + q\omega(E_{14}\otimes E_{41} + E_{41}\otimes E_{14}) - qE_{22}\otimes E_{22} + (q^{-1}-q)E_{22}\otimes E_{33} \\ & + (q^{-1}\lambda^{-2}-q)E_{22}\otimes E_{44} - \omega(E_{23}\otimes E_{32} + E_{32}\otimes E_{23}) \\ & + \lambda^{-1}\omega(E_{24}\otimes E_{42} + E_{42}\otimes E_{24}) - qE_{33}\otimes E_{33} + (q^{-1}\lambda^{-2}-q)E_{33}\otimes E_{44} \\ & + \lambda^{-1}(E_{34}\otimes E_{43} + E_{43}\otimes E_{34}) + q^{-1}\lambda^{-2}E_{44}\otimes E_{44}\ , \end{aligned} \qquad (3.9)$$

$$
\begin{aligned}
b^{-1} \;=\;& q^{-1}\lambda^{-2}E_{11}\otimes E_{11} + q^{-1}\lambda^{-1}(E_{12}\otimes E_{21} + E_{21}\otimes E_{12}) \\
&+ q^{-1}\lambda^{-1}\omega(E_{13}\otimes E_{31} + E_{31}\otimes E_{13}) + q^{-1}\omega(E_{14}\otimes E_{41} + E_{41}\otimes E_{14}) \\
&+ (q^{-1}\lambda^{-2} - q^{-1})E_{22}\otimes E_{11} - q^{-1}E_{22}\otimes E_{22} - \omega(E_{23}\otimes E_{32} + E_{32}\otimes E_{23}) \\
&- q^{-1/2}x^{1/2}\omega(E_{24}\otimes E_{31} + E_{42}\otimes E_{13}) + \lambda\omega(E_{24}\otimes E_{42} + E_{42}\otimes E_{24}) \\
&+ (q^{-1}\lambda^{-2} - q^{-1})E_{33}\otimes E_{11} + (q - q^{-1})E_{33}\otimes E_{22} - q^{-1}E_{33}\otimes E_{33} \\
&+ q^{1/2}x^{1/2}(E_{34}\otimes E_{21} + E_{43}\otimes E_{12}) + \lambda(E_{34}\otimes E_{43} + E_{43}\otimes E_{34}) \\
&+ xE_{44}\otimes E_{11} + (q\lambda^2 - q^{-1})E_{44}\otimes E_{22} + (q\lambda^2 - q^{-1})E_{44}\otimes E_{33} \\
&+ q\lambda^2 E_{44}\otimes E_{44}\;.
\end{aligned} \tag{3.10}
$$

4 A cubic algebra, baxterization and exact solvability

These solutions satisfy the cubic equations

$$
(b_i + q)\left(b_i - q\lambda^2\right)\left(b_i - q^{-1}\lambda^{-2}\right) \;=\; 0\;, \tag{4.1}
$$

$$
\left(b_i^{-1} + q^{-1}\right)\left(b_i^{-1} - q\lambda^2\right)\left(b_i^{-1} - q^{-1}\lambda^{-2}\right) \;=\; 0\;. \tag{4.2}
$$

The explicit expressions for the projectors $\mathcal{O}_a$ can be obtained from (3.9, 3.10) by inverting (3.7, 3.8), i.e.,

$$
\begin{aligned}
\mathcal{O}_0 \;&=\; \frac{[\mu]}{[2\mu][2\mu+1]}\left([\mu+1]\mathrm{Id} + \frac{1}{q-q^{-1}}\left(\lambda b - \lambda^{-1}b^{-1}\right)\right)\;, \\
\mathcal{O}_1 \;&=\; \frac{[\mu][\mu+1]}{[2\mu][2\mu+2]}\left(\left(q\lambda^2 + q^{-1}\lambda^{-2}\right)\mathrm{Id} - b - b^{-1}\right)\;, \\
\mathcal{O}_2 \;&=\; \frac{[\mu+1]}{[2\mu+1][2\mu+2]}\left([\mu]\mathrm{Id} + \frac{1}{q-q^{-1}}\left(-q^{-1}\lambda^{-1}b + q\lambda b^{-1}\right)\right)\;.
\end{aligned} \tag{4.3}
$$

We can use the cubic equations (4.2) in a Baxterisation procedure [20] to get solution of the Yang–Baxter algebra

$$
\begin{aligned}
\check{\mathcal{R}}_{i,i+1}(u)\check{\mathcal{R}}_{i+1,i+2}(u+v)\check{\mathcal{R}}_{i,i+1}(v) \;&=\; \check{\mathcal{R}}_{i+1,i+2}(v)\check{\mathcal{R}}_{i,i+1}(u+v)\check{\mathcal{R}}_{i+1,i+2}(u)\;, \\
\check{\mathcal{R}}_{i,i+1}(u)\check{\mathcal{R}}_{j,j+1}(v) \;&=\; \check{\mathcal{R}}_{j,j+1}(v)\check{\mathcal{R}}_{i,i+1}(u) \qquad \text{for } |i-j| \geq 2\;.
\end{aligned} \tag{4.4}
$$

The matrix $\check{\mathcal{R}}$ is related to the matrix $\mathcal{R}$ by $\check{\mathcal{R}} = \mathcal{P}\mathcal{R}$, the operator $\mathcal{P}$ being the permutation map $\mathcal{P} : x\otimes y \mapsto y\otimes x$.

In the simplest case where b_i satisfies a quadratic relation (Hecke case), it is possible to find a linear combination of b and b^{-1} that is solution of the Yang–Baxter algebra (Baxterisation).

We look here for solutions of the Yang–Baxter algebra (4.4) with $\check{\mathcal{R}}(u)$ in the linear span of Id, b, b^{-1} with coefficients depending on u.

We find the solution

$$
\check{\mathcal{R}}_{i,i+1}(u) = 1 + \frac{1}{x}\left((e^u - 1)b_i + (e^{-u} - 1)b_i^{-1}\right)\;, \tag{4.5}
$$

relying on the fact that b obeys the supplementary relation

$$
\begin{aligned}
0 \; = \; & b_i b_{i\pm1}^{-1} b_i - b_{i\pm1} b_i^{-1} b_{i\pm1} - b_i^{-1} b_{i\pm1} b_i^{-1} + b_{i\pm1}^{-1} b_i b_{i\pm1}^{-1} \\
& - x(b_i b_{i\pm1}^{-1} - b_i^{-1} b_{i\pm1} - b_{i\pm1} b_i^{-1} + b_{i\pm1}^{-1} b_i) \\
& - x(q^{-1}(b_i - b_{i\pm1}) - q(b_i^{-1} - b_{i\pm1}^{-1}))
\end{aligned}
\qquad (4.6)
$$

or equivalently

$$
\begin{aligned}
& (b_i - x) b_{i\pm1}^{-1} (b_i - x) - b_i^{-1} (b_{i\pm1} - x) b_i^{-1} = \\
& \qquad = (b_{i\pm1} - x) b_i^{-1} (b_{i\pm1} - x) - b_{i\pm1}^{-1} (b_i - x) b_{i\pm1}^{-1} \; .
\end{aligned}
\qquad (4.7)
$$

The algebra satisfied by the operators b_i is then given by (1.2-1.5). It is sufficient to define an exactly solvable periodic spin chain. This algebra was already used in [5] to obtain solutions of the Yang–Baxter algebra (4.4).

We notice that we do not have a full BWM algebra: in the algebra generated by b_i, b_i^{-1}, the operators e_i such that

$$e_i^2 = \alpha e_i \qquad (4.8)$$

satisfy neither

$$e_i e_{i\pm1} e_i = \alpha' e_i \qquad (4.9)$$

nor

$$e_i b_{i\pm1} e_i = \alpha'' e_i \; . \qquad (4.10)$$

The relations (1.2–1.5) are nevertheless enough to ensure that the $\check{\mathcal{R}}$-matrix (4.5) satisfies the Yang–Baxter algebra.

The $\check{\mathcal{R}}$-matrix with spectral parameter u satisfies the inversion relation:

$$\check{\mathcal{R}}(u)\check{\mathcal{R}}(-u) = \zeta(u) \; , \qquad (4.11)$$

with

$$\zeta(u) = e^{-2u}(e^u - \lambda^{-2})(e^u - \lambda^2)(e^u - q^2\lambda^{-2})(e^u - q^{-2}\lambda^2)/x^2 \; . \qquad (4.12)$$

It has PT symmetry:

$$\mathcal{R}_{21}(u) \equiv \mathcal{P}\mathcal{R}_{12}(u)\mathcal{P} = \mathcal{R}_{12}(u)^{t_1 t_2} \; . \qquad (4.13)$$

It satisfies also the crossing unitarity property [21, 22]:

$$\mathcal{R}_{12}(u)^{t_1} M_1 \mathcal{R}_{21}(-u-2\rho)^{t_1} M_1^{-1} = \xi(u+\rho) \; , \qquad (4.14)$$

with

$$\rho = \ln q \; , \qquad M = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -q^2 & \\ & & & q^2 \end{pmatrix} \qquad (4.15)$$

and

$$\xi(u) = -(q^{-1}e^u - 1)(1 - qe^{-u})(qe^u - 1)(1 - q^{-1}e^{-u})/x^2 \; . \qquad (4.16)$$

We define the row-to-row transfer matrix on a closed chain as $Tr_0\mathcal{T}(u)$, where $\mathcal{T}(u)$ is the monodromy matrix given by

$$\mathcal{T}(u) = \mathcal{R}_{0L}(u)\mathcal{R}_{0\ L-1}(u)\cdots\mathcal{R}_{01}(u)\ . \tag{4.17}$$

The Yang–Baxter algebra satisfied by $\mathcal{R}$ ensures that transfer matrices with different spectral parameters commute, i.e.

$$[Tr_0\,\mathcal{T}(u), Tr_0\,\mathcal{T}(v)] = 0 \qquad \forall u, v\ . \tag{4.18}$$

From the $\mathcal{R}$-matrix one can extract a spin chain hamiltonian with nearest neighbour interaction

$$\mathcal{H}_{\text{per}} = x\,\frac{d}{du}\bigg|_{u=0}\mathcal{T}(u) = \sum_{i=1}^{L-1}\mathcal{H}_{i\ i+1} + \mathcal{H}_{L\,1}\ , \tag{4.19}$$

with

$$\mathcal{H}_{i,i+1} = x\,\frac{d}{du}\bigg|_{u=0}\check{\mathcal{R}}_{i,i+1}(u) = b_i - b_i^{-1}\ . \tag{4.20}$$

With periodic boundary conditions, this hamiltonian also commutes with all the transfer matrices, which is the requirement for its exact solvability. The hamiltonian with ordinary periodic boundary conditions is however not $\mathcal{U}_q(sl(2|1))$-invariant. A method was developed in [23] to construct a periodic hamiltonian which is still $\mathcal{U}_q(sl(2|1))$-invariant, by adding a "$\mathcal{H}_{L1}$"-type term which is not completely local. A simpler solution is also presented in [24].

5 Two site quantum chain hamiltonian

To obtain a model of interacting electrons, we will use, as in [4] the following interpretation of the states of the representation in terms of fermionic states:

$$|1\rangle = |\uparrow\downarrow\rangle = c_\downarrow^\dagger c_\uparrow^\dagger\,|\emptyset\rangle \qquad |2\rangle = |\downarrow\rangle = c_\downarrow^\dagger\,|\emptyset\rangle \qquad |3\rangle = |\uparrow\rangle = c_\uparrow^\dagger\,|\emptyset\rangle \qquad |4\rangle = |\emptyset\rangle\ . \tag{5.1}$$

We will also use

$$n_\uparrow = c_\uparrow^\dagger c_\uparrow = E_{11} + E_{33}\ , \tag{5.2}$$

$$n_\downarrow = c_\downarrow^\dagger c_\downarrow = E_{11} + E_{22}\ , \tag{5.3}$$

$$n = n_\uparrow + n_\downarrow = 2E_{11} + E_{22} + E_{33}\ . \tag{5.4}$$

(5.5)

The expression of the spin chain hamiltonian obtained in this case is given by

$$\mathcal{H}^{dist} = \mathcal{H}_{hop} + \mathcal{H}_{diag}^{dist}\ , \tag{5.6}$$

where

$$\begin{aligned}
\mathcal{H}_{hop} = & \left(c^\dagger_{\uparrow i+1}c^\dagger_{\downarrow i+1}c_{\downarrow i}c_{\uparrow i} + c^\dagger_{\uparrow i}c^\dagger_{\downarrow i}c_{\downarrow i+1}c_{\uparrow i+1}\right) \\
& + \left(c^\dagger_{\uparrow i+1}c_{\uparrow i} + c^\dagger_{\uparrow i}c_{\uparrow i+1}\right)\Big\{ -[\mu] + n_{\downarrow i}\left([\mu]+q^{-1/2}y\right) + n_{\downarrow i+1}\left([\mu]-q^{1/2}y\right) \\
& \qquad + n_{\downarrow i}n_{\downarrow i+1}\left(-[\mu]+[\mu+1]+(q^{1/2}-q^{-1/2})y\right)\Big\} \\
& + \omega\left(c^\dagger_{\downarrow i+1}c_{\downarrow i} + c^\dagger_{\downarrow i}c_{\downarrow i+1}\right)\Big\{ -[\mu] + n_{\uparrow i}\left([\mu]-q^{1/2}y\right) + n_{\uparrow i+1}\left([\mu]+q^{-1/2}y\right) \\
& \qquad + n_{\uparrow i}n_{\uparrow i+1}\left(-[\mu]+[\mu+1]+(q^{1/2}-q^{-1/2})y\right)\Big\} \qquad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}^{dist}_{diag} = & \; n_{\uparrow i}n_{\downarrow i} + n_{\uparrow i+1}n_{\downarrow i+1} - [2\mu+1] \\
& + q^{\mu+1}[\mu](n_{\uparrow i}+n_{\downarrow i}) + q^{-\mu-1}[\mu](n_{\uparrow i+1}+n_{\downarrow i+1}) \; , \qquad (5.8)
\end{aligned}$$

where μ is related to the parameter of the representation λ by $\lambda = q^\mu$. By construction, the creation and annihilation operators on different sites commute. A Jordan–Wigner transformation can restore the standard anticommutation property.

This exactly solvable hamiltonian with two parameters $\lambda = q^\mu$ and q was already considered in [9, 4]. In [4], it was obtained as the derivative of the spectral parameter $\check{\mathcal{R}}$-matrix of the four dimensional representation of $\mathcal{U}_q(sl(sl2|1))$. The eigenstates of the periodic model are found in [25] using the algebraic Bethe ansatz.

6 Reflection equations and open chain hamiltonian

6.1 Reflection equations

We can also get an exactly solvable and $\mathcal{U}_q(sl(2|1))$-invariant *open* chain hamiltonian by solving the reflection equations [26, 27, 28, 22, 29]

$$\mathcal{R}_{12}(u-v)\mathcal{K}^-_1(u)\mathcal{R}_{21}(u+v)\mathcal{K}^-_2(v) = \mathcal{K}^-_2(v)\mathcal{R}_{12}(u+v)\mathcal{K}^-_1(u)\mathcal{R}_{21}(u-v) \qquad (6.1)$$

and

$$\begin{aligned}
& \mathcal{R}_{12}(-u+v)\mathcal{K}^+_1(u)^{t_1}M_1^{-1}\mathcal{R}_{21}(-u-v-2\rho)M_1\mathcal{K}^+_2(v)^{t_2} = \\
& \qquad \mathcal{K}^+_2(v)^{t_2}M_1\mathcal{R}_{12}(-u-v-2\rho)M_1^{-1}\mathcal{K}^+_1(u)^{t_1}\mathcal{R}_{21}(-u+v) \; . \qquad (6.2)
\end{aligned}$$

The simplest solution for these equations is [29]

$$\mathcal{K}^-(u) = \mathrm{Id} \qquad \text{and} \qquad \mathcal{K}^+(u) = M \; . \qquad (6.3)$$

This is always a solution when the spectral parameter $\mathcal{R}$-matrix is obtained via self-Baxterisation [20], i.e. when the $\check{\mathcal{R}}$-matrix belongs to the algebra generated by b_i, since in this case $\check{\mathcal{R}}$-matrices with different spectral parameters commute:

$$\left[\check{\mathcal{R}}(u), \check{\mathcal{R}}(v)\right] = 0 \qquad \forall u, v \in \mathbb{C} \; . \qquad (6.4)$$

The matrix M may in this case be interpreted as a Markov trace, as in [30].

More generally, there are two diagonal one parameter solutions for $\mathcal{K}^-(u)$ (up to an overall function of u), given by

$$\mathcal{K}_a^-(u) = \frac{1}{(1+C)(1+q^2C)} \times \\ \times \begin{pmatrix} (e^{-u}+C)(e^{-u}+q^2C) & & & \\ & (e^{u}+C)(e^{-u}+q^2C) & & \\ & & (e^{u}+C)(e^{-u}+q^2C) & \\ & & & (e^{u}+C)(e^{u}+q^2C) \end{pmatrix} \tag{6.5}$$

and

$$\mathcal{K}_b^-(u) = \frac{1}{1+C} \begin{pmatrix} e^{-u}+C & & & \\ & e^{-u}+C & & \\ & & e^{u}+C & \\ & & & e^{u}+C \end{pmatrix} . \tag{6.6}$$

Solutions for $\mathcal{K}^+(u)$ are given by [29]

$$K^+(u) = K^-(-u-\rho)^t M \ . \tag{6.7}$$

Note that the number of one parameter diagonal solutions is the same as for the supersymmetric t–J model [10] and is equal to the rank of the underlying algebra.

6.2 Open chain transfer matrix and exactly solvable hamiltonian

Using the Reflection Equations (6.1), (6.2), and the Yang–Baxter algebra (4.4), one can prove that the double-row transfer matrices $t(u)$ [22]

$$\begin{aligned} t(u) \ &= \ \zeta(u)^{-L} \mathrm{tr}\, \mathcal{K}^+(u) T(u) \mathcal{K}^-(u) T(-u)^{-1} \qquad (6.8) \\ &= \ \mathrm{tr}_0\, \mathcal{K}_0^+(u) \check{\mathcal{R}}_{L0}(u) \check{\mathcal{R}}_{L-1,L}(u) \cdots \check{\mathcal{R}}_{23}(u) \check{\mathcal{R}}_{12}(u) \times \\ &\quad \times \ \mathcal{K}_1^-(u) \check{\mathcal{R}}_{12}(u) \cdots \check{\mathcal{R}}_{23}(u) \cdots \check{\mathcal{R}}_{L-1,L}(u) \check{\mathcal{R}}_{L0}(u) \qquad (6.9) \end{aligned}$$

commute for different values of u [27, 28, 29, 31].

We then compute

$$\left.\frac{dt(u)}{du}\right|_{u=0} - \left.\frac{d}{du} \mathrm{tr}_0\, \mathcal{K}_0^+(u)\right|_{u=0} = \\ = \left(\mathrm{tr}_0\, \mathcal{K}_0^+(0)\right) \left(2 \sum_{j=1}^{L-1} \mathcal{H}_{j,j+1} + \left.\frac{d}{du} \mathcal{K}_1^-(u)\right|_{u=0} \right) + 2 \mathrm{tr}_0\, \mathcal{K}_0^+(0) \mathcal{H}_{L0} \ . \tag{6.10}$$

It is standard to use this expression, divided by $\mathrm{tr}_0\, \mathcal{K}_0^+(0)$, to get a spin chain hamiltonian with nearest neighbour interaction. By construction, this hamiltonian commutes with $t(u)$ for all values of u and it is hence exactly solvable [27].

This operation however provides nothing here, since, for all the diagonal solutions for $\mathcal{K}^+$, we have $\mathrm{tr}_0\,\mathcal{K}_0^+(0) = 0$. This phenomenon was noticed in [31], and explained by the use of typical representations, which implies $\mathrm{tr}\,M = 0$ (actually $\mathrm{Str}\,M = 0$ if no bosonisation is performed). A method was found there to prove that, in the case

$$\mathcal{K}^-(u) = 1 \qquad \text{and} \qquad \mathcal{K}^+(u) = M \ , \tag{6.11}$$

the quantum chain hamiltonian

$$\sum_{j=1}^{L-1} \mathcal{H}_{j,j+1} \tag{6.12}$$

still commuted with $t(u)$ for all values of u. The $\mathcal{U}_q(sl(2|1))$ symmetry is built-in in this case, since the expression of the hamiltonian (6.12) contains only the coproduct of some Casimir operators (See equations (4.20), (3.7), (3.8) and (3.3) which provide the expression of $\mathcal{H}_{i,i+1}$ in terms of some $(\rho\otimes\rho)\Delta(\mathcal{C}_p)$). This hamiltonian is then both exactly solvable and quantum group invariant.

An other way to obtain an hamiltonian with local interaction in the cases when $\mathrm{tr}_0\,\mathcal{K}_0^+(0) = 0$ is to take the second derivative of $t(u)$ at $u = 0$. This method was also used in [32], where the vanishing of the factor was due to the fact that q was such that $q^4 = 1$. It applies also with the solutions for $\mathcal{K}^+$ different from M and given by (6.7) and (6.5) or (6.6).

$$\begin{aligned}\left.\frac{d^2t(u)}{du^2}\right|_{u=0} = &\left(2\frac{d}{du}\mathrm{tr}_0\,\mathcal{K}_0^+(u)\Big|_{u=0} + 4\mathrm{tr}_0\left(\mathcal{K}_0^+(0)\mathcal{H}_{L0}\right)\right)\left(2\sum_{j=1}^{L-1}\mathcal{H}_{j,j+1} + \frac{d}{du}\mathcal{K}_1^-(u)\Big|_{u=0}\right)\\ &+A_1 + A_2 + A_3 + A_4 \ ,\end{aligned} \tag{6.13}$$

where

$$A_1 = \left.\frac{d^2}{du^2}\mathrm{tr}_0\,\mathcal{K}_0^+(u)\right|_{u=0} \ , \tag{6.14}$$

$$A_2 = 4\mathrm{tr}_0\left(\frac{d}{du}\mathcal{K}_0^+(u)\Big|_{u=0}\mathcal{H}_{L0}\right) \ , \tag{6.15}$$

$$A_3 = 2\mathrm{tr}_0\,\mathcal{K}_0^+(0)\left.\frac{d^2}{du^2}\check{\mathcal{R}}_{L0}(u)\right|_{u=0} \ , \tag{6.16}$$

$$A_4 = 2\mathrm{tr}_0\left(\mathcal{K}_0^+(0)\mathcal{H}_{L0}\mathcal{H}_{L0}\right) \ . \tag{6.17}$$

Now the factor $\left(2\frac{d}{du}\mathrm{tr}_0\,\mathcal{K}_0^+(u)\big|_{u=0} + 4\mathrm{tr}_0\left(\mathcal{K}_0^+(0)\mathcal{H}_{L0}\right)\right) = 2\,\frac{d}{du}\mathrm{tr}_0\,\mathcal{K}_0^+(u)\check{\mathcal{R}}_{L0}^2\big|_{u=0}$ in front of the hamiltonian of interest can be chosen to be non-zero. Moreover, it is proportional to the identity, so that we can use

$$\left(4\,\frac{d}{du}\mathrm{tr}_0\,\mathcal{K}_0^+(u)\check{\mathcal{R}}_{L0}^2\Big|_{u=0}\right)^{-1}\left.\frac{d^2t(u)}{du^2}\right|_{u=0} \tag{6.18}$$

as a spin chain hamiltonian with nearest neighbour interaction.

The term $\frac{d}{du}\mathcal{K}_1^-(u)\big|_{u=0}$ contributes to a boundary term on site 1.

The term A_1 obviously contributes only as constant. The terms A_2, A_3 and A_4 contribute to boundary terms on the last site L of the chain. Note that the sum $A_1 + A_2 + A_3 + A_4$ is equal to

$$A_1 + A_2 + A_3 + A_4 = \frac{d^2}{du^2} \mathrm{tr}_0 \, \mathcal{K}_0^+(u) \check{\mathcal{R}}_{L0}^2 \Big|_{u=0} . \qquad (6.19)$$

The expression of the exactly solvable hamiltonian with open boundary condition is then

$$\mathcal{H}_{open} = \sum_{j=1}^{L-1} \mathcal{H}_{j,j+1} + \frac{1}{2} \frac{d}{du} \mathcal{K}_1^-(u) \Big|_{u=0} + \frac{\frac{d^2}{du^2} \mathrm{tr}_0 \, \mathcal{K}_0^+(u) \check{\mathcal{R}}_{L0}^2 \Big|_{u=0}}{4 \, \frac{d}{du} \mathrm{tr}_0 \, \mathcal{K}_0^+(u) \check{\mathcal{R}}_{L0}^2 \Big|_{u=0}} . \qquad (6.20)$$

From the expressions of the boundary terms in (6.20), one can prove that, if the solution of the reflections equations are multiplied by arbitrary functions of u, the hamiltonian is left unchanged (up to constant terms).

6.3 Integrable boundary terms

We use the construction of section 5 for the expression of the bulk term $\mathcal{H}_{j,j+1} = \mathcal{H}_{j,j+1}^{dist}$ of Eq. (5.6), (5.7) and (5.8). We then include the results of section 6 for the boundary terms (inserting the $\check{\mathcal{R}}$ matrix of section 4). We get

$$\mathcal{H}_{open}^{dist} = \sum_{j=1}^{L-1} \mathcal{H}_{j,j+1}^{dist} + \mathcal{B}_1 + \mathcal{B}_L . \qquad (6.21)$$

The boundary term $\mathcal{B}_1 = \frac{d}{du} \mathcal{K}_1^-(u) \big|_{u=0}$ on site 1 takes one of the forms

$$\mathcal{B}_1^0 = 0 \qquad \text{(in the case } \mathcal{K}^- = 1) \qquad (6.22)$$

or

$$\mathcal{B}_1^a = \frac{-1}{(1+C_-)(1+q^2 C_-)} \times \times \Big\{ (2 + C_- + q^2 C_-) E_{11} + (1 + C_-) \left(E_{22} + E_{33} \right) \Big\} \qquad (6.23)$$

or

$$\mathcal{B}_1^b = \frac{-1}{(1+C_-)} \left(E_{11} + E_{22} \right) , \qquad (6.24)$$

(mutually exclusive) depending on the choice of the solution ($\mathcal{K}_a^-$ or $\mathcal{K}_b^-$) for the matrix $\mathcal{K}^-$. It depends on the parameter $C_- \equiv C$ from (6.5) or (6.6).

These expressions read, in terms of number of particles

$$\mathcal{B}_1^0 = 0 , \qquad (6.25)$$

$$\mathcal{B}_1^a = \frac{-1}{(1+C_-)(1+q^2 C_-)} \Big\{ (q^2 - 1) C_- n_{\uparrow 1} n_{\downarrow 1} + (1 + C_-) \left(n_{\uparrow 1} + n_{\downarrow 1} \right) \Big\} , \qquad (6.26)$$

$$\mathcal{B}_1^b = \frac{-1}{(1+C_-)} n_{\downarrow 1} . \qquad (6.27)$$

The boundary term $\mathcal{B}_L = \frac{\left.\frac{d^2}{du^2}\mathrm{tr}_0\, \mathcal{K}_0^+(u)\check{\mathcal{R}}_{L0}^2\right|_{u=0}}{\left.4\,\frac{d}{du}\mathrm{tr}_0\, \mathcal{K}_0^+(u)\check{\mathcal{R}}_{L0}^2\right|_{u=0}}$ on site L takes one of the forms

$$\mathcal{B}_L^0 = 0 \qquad \text{(in the case } \mathcal{K}^+ = M\text{)} \tag{6.28}$$

or

$$\begin{aligned}\mathcal{B}_L^a &= \frac{1}{(1+q^{-1}\lambda^{-2}C_+)(1+q\lambda^{-2}C_+)} \times \\ &\times\Big\{(2+q^{-1}\lambda^{-2}C_+q^{-1}\lambda^{-2}C_+)E_{11} + (1+q\lambda^{-2}C_+)\left(E_{22}+E_{33}\right)\Big\}\end{aligned} \tag{6.29}$$

or

$$\mathcal{B}_L^b = \frac{1}{(1+q^{-1}\lambda^{-2}C_+)}\left(E_{11}+E_{22}\right)\ , \tag{6.30}$$

depending on the choice of solution for the matrix $\mathcal{K}^+$ (which is independent of the choice for $\mathcal{K}^-$). It depends on a parameter C_+ coming from (6.5) or (6.6) when used as solutions for $\mathcal{K}^+$ given by (6.7).

These expressions read, in terms of number of particles and after a redefinition of the parameter C_+ that eliminates the dependence in λ,

$$\mathcal{B}_L^0 = 0\ , \tag{6.31}$$

$$\mathcal{B}_L^a = \frac{1}{(1+C'_+)(1+q^2C'_+)}\Big\{(1-q^2)C'_+ n_{\uparrow L}n_{\downarrow L} + (1+q^2C'_+)\left(n_{\uparrow L}+n_{\downarrow L}\right)\Big\}\ , \tag{6.32}$$

$$\mathcal{B}_L^b = \frac{1}{(1+C'_+)}n_{\downarrow L}\ . \tag{6.33}$$

As we will see in the next section, there exists a non trivial choice for the boundary terms $\mathcal{B}_1^b$ and $\mathcal{B}_L^b$ that leads to an exactly solvable hamiltonian *with* $\mathcal{U}_q(sl(2|1))$ *invariance.*

7 Another spin chain hamiltonian: using the fermionic basis of $\mathcal{U}_q(sl(2|1))$

Alternatively, we could have used form the beginning the fermionic basis to describe the quantum algebra. In this basis, the Cartan matrix is

$$(a_{ij})_{ferm} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\ . \tag{7.1}$$

The generators K_1, K_2, E_1, E_2, F_1, F_2 in the fermionic basis are, in terms of the generators in the distinguished basis:

$$\begin{aligned} K_1 = k_1^{-1}k_2^{-1} &\quad K_2 = k_2 \\ E_1 = e_3 &\quad E_2 = f_2k_2^{-1} \\ F_1 = -f_3 &\quad F_2 = k_2e_2\ . \end{aligned} \tag{7.2}$$

As algebras, $\mathcal{U}_q(sl(2|1))$ in both bases are identical. Only the choices of simple root are different. However, the Hopf structure are not identical: the coproduct in the fermionic basis is given by

$$\begin{aligned}\tilde{\Delta}(K_i) &= K_i \otimes K_i \ , \\ \tilde{\Delta}(E_i) &= E_i \otimes 1 + K_i \otimes E_i \ , \\ \tilde{\Delta}(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i \ , \end{aligned} \tag{7.3}$$

which, in terms of the distinguished generators, is different from (2.4) (See [8]), and will produce (using the same algorithm as for the distinguished case) a quantum chain hamiltonian different from (5.6):

$$\mathcal{H}^{ferm} = \mathcal{H}_{hop} + \mathcal{H}_{diag}^{ferm} \tag{7.4}$$

with

$$\begin{aligned}\mathcal{H}_{diag}^{ferm} &= n_{\uparrow i} n_{\downarrow i} + n_{\uparrow i+1} n_{\downarrow i+1} - [2\mu + 1] \\ &\quad + q^{\mu+1}[\mu](n_{\uparrow i} + n_{\downarrow i+1}) + q^{-\mu-1}[\mu](n_{\uparrow i+1} + n_{\downarrow i}) \ . \end{aligned} \tag{7.5}$$

The hamiltonians obtained with the distinguished basis and with the fermionic basis are actually very close to each other: the only difference is in boundary terms, which are symmetric in $\uparrow$ and $\downarrow$ in the distinguished case, but not in the fermionic one. When summed over the chain, the difference of the hamiltonians $\mathcal{H}_{open}^{ferm}$ and $\mathcal{H}_{open}^{dist}$ (without integrable boundary terms added) is indeed

$$\mathcal{H}_{open}^{ferm} - \mathcal{H}_{open}^{dist} = \sum_{j=1}^{L-1} \left(\mathcal{H}_{diag\ j,j+1}^{ferm} - \mathcal{H}_{diag\ j,j+1}^{dist} \right) = \frac{x}{q - q^{-1}} (n_{\downarrow L} - n_{\downarrow 1}) \ . \tag{7.6}$$

The hamiltonian $\mathcal{H}_{open}^{ferm}$ is actually *equal* to the hamiltonian (6.21) obtained with the distinguished basis, now including the integrable boundary terms $\mathcal{B}_1^b$ (6.27) and $\mathcal{B}_L^b$ (6.33) coming from the *second* solution (6.6) of the reflection equations (6.1), (6.2), for the particular choice of parameters

$$C_- = \frac{q - q^{-1}}{x} - 1 \qquad C_+ = q\lambda^2 C'_+ = q\lambda^2 \left(\frac{q - q^{-1}}{x} - 1 \right) \ . \tag{7.7}$$

Although different, the two Hopf structures defined by (2.4) and (7.3) are equivalent [33] through a Reshetikhin twist [34]

$$\tilde{\Delta}(a) = \mathcal{F} \Delta(a) \mathcal{F}^{-1} \ , \tag{7.8}$$

satisfying

$$(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1 \ , \tag{7.9}$$

$$(\mathcal{F} \otimes 1)(\Delta \otimes 1)\mathcal{F} = (1 \otimes \mathcal{F})(1 \otimes \Delta)\mathcal{F} \ . \tag{7.10}$$

It was indeed proved in [33] that an operator $\mathcal{F}$ satisfying (7.8) could be obtained as the factor of the universal $\mathcal{R}$-matrix of $\mathcal{U}_q(sl(2|1))$ related with the fermionic root which defines the super-Weyl reflection that relates the two bases.

This implies that open quantum chains built with the two-site hamiltonians (5.6) and (7.4) are equivalent, the equivalence matrix being given by

$$\Big(\rho\otimes\cdots\otimes\rho\Big)\mathcal{F}^{(L)}\ , \tag{7.11}$$

$\mathcal{F}^{(L)}$ being defined recursively as

$$\mathcal{F}^{(L)}\equiv(\mathcal{F}\otimes 1^{\otimes^{L-1}})(\Delta\otimes 1^{\otimes^{L-1}})\mathcal{F}^{(L-1)}\ . \tag{7.12}$$

As in [8], this equivalence is simple for the two site hamiltonians (i.e. for (5.6) and (7.4) themselves). However, it becomes highly non trivial for longer chains, the reason being that the equivalence produced by the twist is non local.

In [35], Reshetikhin twists are applied to the supersymmetric t–J model and to the supersymmetric Hubbard model with pair hopping (5.6). This leads to multiparametric hamiltonians. The effects of these twists are visible in the bulk term of the hamiltonian, in contrast with the action of our twist which relates the distinguished construction to the fermionic one, and which affects only boundary terms.

8 Another example

We can also obtain $\mathcal{U}_q(sl(2))\otimes U(1)$ invariant Hamitonians as

$$\mathcal{H}=\sum_{i=1}^{L-1}1\otimes\cdots\otimes\underbrace{(\rho\otimes\rho)\Delta(\mathrm{Pol}\{\mathcal{Q}_p^{(+)},\mathcal{Q}_p^{(-)}\})}_{\text{sites }i,i+1}\otimes\cdots\otimes 1\ . \tag{8.1}$$

Choosing the four dimensional representation with the fixed parameter $\lambda=q^{-1/2}$, and taking a polynomial in $\mathcal{Q}^{(+)}$ only, we get for instance

$$\begin{aligned}\mathcal{H}^{TL}_{i,i+1} &= c^\dagger_{\uparrow i+1}c^\dagger_{\downarrow i+1}c_{\downarrow i}c_{\uparrow i}+c^\dagger_{\uparrow i}c^\dagger_{\downarrow i}c_{\downarrow i+1}c_{\uparrow i+1}-S^+_iS^-_{i+1}-S^-_iS^+_{i+1}\\ &+\Big(c^\dagger_{\uparrow i+1}c_{\uparrow i}-c^\dagger_{\uparrow i}c_{\uparrow i+1}\Big)\,\omega\left\{q^{-1}n_{\downarrow i}+qn_{\downarrow i+1}-(q+q^{-1})n_{\downarrow i}n_{\downarrow i+1}\right\}\\ &+\Big(-c^\dagger_{\downarrow i+1}c_{\downarrow i}+c^\dagger_{\downarrow i}c_{\downarrow i+1}\Big)\left\{n_{\uparrow i}+n_{\uparrow i+1}-2n_{\uparrow i}n_{\uparrow i+1}\right\}\\ &+(n_{\uparrow i}-n_{\uparrow i+1})\left\{q^{-1}n_{\downarrow i}-qn_{\downarrow i+1}-(q-q^{-1})n_{\downarrow i}n_{\downarrow i+1}\right\}\ ,\end{aligned} \tag{8.2}$$

which satisfies the Temperley–Lieb algebra

$$b_i^2 = 0 \tag{8.3}$$

$$b_ib_{i\pm1}b_i = b_i \tag{8.4}$$

$$b_ib_j = b_jb_i \qquad \text{for} \qquad |i-j|\geq 2\ . \tag{8.5}$$

Such hamiltonians were found in [36, 19]. It was noticed that, although not hermitian, they lead to hermitian hamiltonian when multiplied by $(1 - 2n_{\downarrow i} - 2n_{\uparrow i} + 4n_{\uparrow i}n_{\downarrow i})$ (the parity operator on one site), the result satisfying also a Temperley–Lieb algebra (with non vanishing square).

It could also be of interest to investigate the use of the hamiltonian (8.2) itself for reaction-diffusion processes [37].

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A Appendix: scasimirs of $\mathcal{U}(sl(2|1))$

We give in this appendix the expressions of the scasimirs of non-deformed superalgebra $\mathcal{U}(sl(2|1))$.

The scasimir of $osp(2|1)$ appeared in [38, 39, 40]. In [40], the expression of the scasimir is also given in the q-deformed case.

The proof of existence of scasimir operators for $osp(1|2n)$ was given in [41, 42], where it was also proved that the scasimir was the square root of a Casimir element of degree $2n$. An explicit expression of the scasimir is written in [42].

The existence of scasimir operators in the case of $sl(m|n)$ is known to Musson [43].

The classical superalgebra $sl(2|1)$ is defined by the relations

$$\begin{aligned}
&[h_1, h_2] = 0 \ , && \\
&[h_i, e_j] = a_{ji} e_j \ , && [h_i, f_j] = -a_{ji} f_j \ , \\
&[e_1, f_1] = h_1 \ , && [e_2, f_2]_+ = h_2 \ , \\
&[e_1, f_2] = [e_2, f_1] = 0 \ , && \\
&[e_2, e_2]_+ = [f_2, f_2]_+ = 0 \ , && \\
&[e_1, e_3] = [f_1, f_3] = 0 \ , &&
\end{aligned} \tag{A.1}$$

where

$$e_3 = [e_1, e_2] \qquad \text{and} \qquad f_3 = [f_2, f_1] \ . \tag{A.2}$$

The last relations in (A.1) may also be written as Serre relations

$$\begin{aligned}
&e_1^2 e_2 - 2 e_1 e_2 e_1 + e_2 e_1^2 = 0 \ , \\
&f_1^2 f_2 - 2 f_1 f_2 f_1 + f_2 f_1^2 = 0 \ .
\end{aligned} \tag{A.3}$$

We define the elements $\mathcal{Q}_p^{(\pm)}$ of the non-quantum $\mathcal{U}(sl(2|1))$ as

$$\begin{aligned}\mathcal{Q}^{(+)}{}_p &= \Big\{h_2(h_1+h_2+1) - f_1e_1 - f_2e_2(h_1+h_2+1) - f_3e_3(h_2-1) \\ &\quad + f_1f_2e_3 + f_3e_2e_1 + f_2f_3e_3e_2\Big\}(-h_1-2h_2-1)^{p-2} \\ &\quad + f_2f_3e_3e_2(-h_1-2h_2+1)^{p-2}\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}\mathcal{Q}^{(-)}{}_p &= \Big\{f_2e_2(h_1+h_2) + f_3e_3(h_2-2) \\ &\quad - f_1f_2e_3 - f_3e_2e_1 - 2f_2f_3e_3e_2\Big\}(-h_1-2h_2)^{p-2}\,,\end{aligned} \tag{A.5}$$

for $p \geq 2$. Their sum $\mathcal{C}_p$ and difference $\mathcal{S}_p$ are, respectively, Casimir operators and scasimirs of $\mathcal{U}(sl(2|1))$, i.e. they satisfy the classical analogues of (2.14, 2.15). The relations (2.8, 2.9, 2.10,2.16, 2.17, 2.18) are still valid as long as the indices p_i are greater or equal to 2. Notice that the classical operators $\mathcal{Q}_p^{(\pm)}$, $\mathcal{C}_p$ and $\mathcal{S}_p$ are not the limits as q goes to 1 of the corresponding quantum ones, but rather limits of some linear combinations of them (See [8]).

Discussions with M. Bauer and V. Lafforgue led to an expression of $\mathcal{S}_2$ in terms of antisymmetrized products of fermionic operators e_i, f_i, $i = 2, 3$ only, as for $osp(1|2n)$ in [42]. This seems to be possible for more general superalgebras.

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